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GAS-DYNAMIC BANK CONTROL OF A SPACECRAFT
IN THE ATMOSPHERE

R. V. Studnev

Translation of: "Gazodinamicheskoye
upravleniye dvizheniyem kosmicheskogo
apparata po krenu v atmosfere", Uprav-
leniye kosmicheskimi apparatami i
korablyami (Control of Spacecraft and
Space Vehicles), Edited by B.N. Petrov
and I.S. Ukolov, Moscow, "Nauka" Press,
1971, Transactions of the Second In-
ternational Symposium of IFAC for
Automatic Control in the Peaceful Use
of Outer Space, Vienna, Austria, Sept-
ember, 1967, pp. 482-496.

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R. V. Studnev (USSR)

ABSTRACT. The Pontryagin maximum principle is used to solve the problem of optimal spacecraft attitude control in the atmosphere. The influence of aerodynamic forces on the spacecraft stability is considered.

At the present time one of the most timely problems is that /482*
of how to control a spacecraft, possessing aerodynamic quality, during its atmosphere entry. Numerous papers discuss the atmospheric re-entry by a spacecraft which is balanced in the sense that its angle of attack is constant, and whose trajectory is controlled by changing the bank angle [1, 2]. In this connection a problem arises of estimating the dynamic possibilities of the motion of a spacecraft relative to its center of mass when compensating for perturbations of the angle of attack and slippage (α, β), and controlling the bank angle. There are many articles devoted to an analysis of the optimal control of a spacecraft's attitude in a vacuum [3 - 6]. A majority of these problems are solved using the Pontryagin maximum principle [7, 8]. Below the maximum principle is used to solve a simplified and analogous problem of the optimal spacecraft attitude control in the atmosphere. The problem is a little more complicated than the problem of motion in a vacuum, since it is necessary to consider the influence of the aerodynamic forces on the stability of the craft.

*Numbers in the margin indicate pagination in the original foreign text.

1

1. Equations of Motion of a Spacecraft Relative to its Center of Mass During Atmosphere Entry

The motion of a spacecraft relative to its center of mass will be considered approximately by neglecting the interaction of the motion of the center of mass. It will also be assumed that, during the time it takes for the bank turns and compensations of deviations in α and β to be completed, the parameters of motion of the craft (V, q) change very little and the equations of motion may be considered to have "frozen" coefficients. Finally, the motion of the craft relative to its center of mass will be assumed to be so slow that it will be possible to neglect in the equations of motion the nonlinear terms such as

Under these assumptions, the equations written relative to the principal central axes of inertia $(OX_1Y_1Z_1)$ (Figure 1) will have the form

$$
\begin{aligned}
\dot{\omega}_z &= \overline{M}_z^\alpha \alpha + u_z, & \dot{\alpha} &+ \omega_z, \\
\dot{\omega}_{y_1} &= \overline{M}_y^\beta \beta + u_y, & \dot{\beta} &= \cos\alpha_0 \cdot \omega_y + \sin\alpha_0 \cdot \omega_x, \\
\dot{\omega}_{x_1} &= \overline{M}_x^\beta \beta + u_x, & \dot{\gamma} &= \cos\alpha_0 \cdot \omega_x - \sin\alpha_0 \cdot \omega_y,
\end{aligned}
$$

$$\tag{1.1}$$
$$\tag{1.2}$$

where α_0 is the balancing angle of attack of the spacecraft, α_0 = const); u_x, u_y, u_z are the moments from the control surfaces divided by the corresponding moments of inertia.

In Equations (1.1) and (1.2) only the moments of the aerodynamic stability of the craft have been retained, since at hypersonic speeds the effect of the aerodynamic damping may be neglected. Equations (1.1) and (1.2) imply that the equations of the three-dimensional motion of a spacecraft can be separated into the equations of longitudinal and lateral motion, (1.1) and (1.2), each of which may be studied separately.
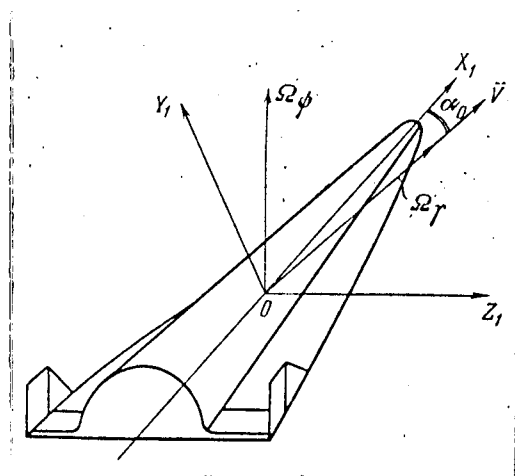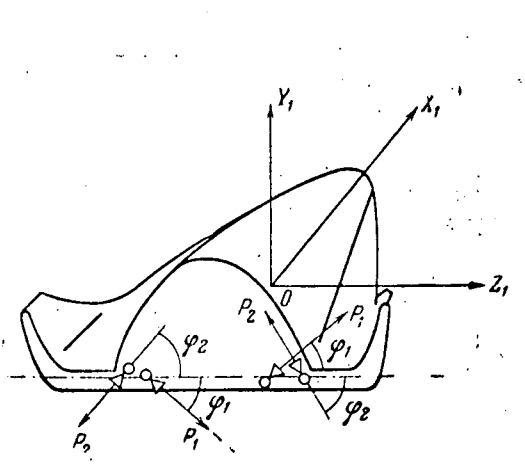
Figure 1.

Figure 2

Let us consider the equations of the lateral motion of a
spacecraft (1.2). Let us transform these equations in such a
way that the banking motion (rotation about the velocity vector
$\overline{V}$) and the yawing motion (variation of the angle β) of the craft
will be separated. Let us multiply the first and second equa-
tions in (1.2) by cos α_0 and sin α_0, respectively, and add them
together. This will produce an equation in $\dot{\Omega}_\psi$. When they are
multiplied by sin α_0 and cos α_0, respectively, and added to-
gether, we obtain an equation in $\dot{\Omega}_\gamma$:

$$\dot{\Omega}_\psi = \sigma_\beta\beta + u_\psi, \qquad \dot{\beta} = \Omega_\psi,$$
$$\dot{\Omega}_\gamma = \sigma_\gamma\beta + u_\gamma, \qquad \dot{\gamma} = \Omega_\gamma. \qquad (1.3)$$

In Equations (1.3) we use the following notation:

$$\Omega_\psi = \cos\alpha_0 \cdot \omega_y + \sin\alpha_0 \cdot \omega_x, \qquad (1.4)$$
$$\Omega_\gamma = \cos\alpha_0 \cdot \omega_x - \sin\alpha_0 \cdot \omega_y,$$

$$\sigma_\beta = \overline{M}_y^\beta \cos\alpha_0 + \overline{M}_x^\beta \sin\alpha_0,$$
$$\sigma_\gamma = \overline{M}_x^\beta \cos\alpha_0 - \overline{M}_y^\beta \sin\alpha_0; \qquad (1.5)$$

$$u_\gamma = u_x \cos\alpha_0 - u_y \sin\alpha_0; \qquad (1.6)$$
$$u_\psi = u_y \cos\alpha_0 + u_x \sin\alpha_0.$$

3

Equations (1.3) imply that the motion of a spacecraft along the angle β does not depend on the angle γ and the control u_γ.

Let us consider a choice of the surfaces controlling u_x, u_y, such that u_γ and u_ψ will be independent. This can be accomplished either by a coordinated deflection of the aerodynamic surfaces (u_x, u_y), or by a special orientation of the control jet engines in the case of gas-dynamic control of the spacecraft.

Let us explain why it is necessary for the spacecraft to position the control jet engines in such a way that one pair will produce the moment u_γ alone, and the second pair — the moment u_ψ alone.

A control jet engine of thrust P_1 whose vector lies in the plane parallel to the plane OY_1Z_1, placed at the tail of the craft at the angle ϕ_1 to the OZ_1 axis, produces reduced moments relative to the OX_1 and OY_1 axes that are given by the formulas (Figure 2)

$$U_{x_1} = 2\frac{P_1 \sin\varphi_1 l_x}{I_x}, \qquad u_{y_1} = 2\frac{P_1 \cos\varphi_1 l_y}{I_y} \qquad (1.7)$$

where l_x, l_y are the distances of one control jet engine to the /484 OX_1 and OY_1 axes, respectively; I_x, I_y are the principal moments of inertia of the spacecraft relative to the OX_1 and OY_1 axes.

Let the angle at which the control jet engine is placed, ϕ_1, be found from the condition that $u_\gamma \equiv 0$ for $P_1 \neq 0$. Substituting the expressions in (1.7) in Equations (1.6), and equating the first relation in (1.6) to zero, we obtain the following condition for ϕ_1

$$\varphi_1 = \operatorname{arctg}\left(\frac{l_y I_x}{l_x I_y} \operatorname{tg}\alpha_0\right). \qquad (1.8)$$

With such a choice of ϕ_1 the control of u_ψ is given by the formula

$$u_\psi = \frac{2P_1 l_y}{I_y} \frac{1}{\cos\alpha_0 \sqrt{1 + \left(\frac{l_y I_x}{l_x I_y} \operatorname{tg}\alpha_0\right)^2}} \cdot \qquad (1.9)$$

In the particular case when $l_y I_x / l_x I_y = 1$ the expression for u_ψ simplifies

$$u_\psi = \frac{2P_1 l_y}{I_y} \cdot \qquad (1.10)$$

Similarly, from the condition $u_\Psi = 0$, we find the orientation of the second pair of control jet engines (Figure 2)

$$\varphi_2 = -\operatorname{arctg}\left(\frac{l_y I_x}{l_x I_y}\frac{1}{\operatorname{tg}\alpha_0}\right) \cdot \qquad (1.11)$$
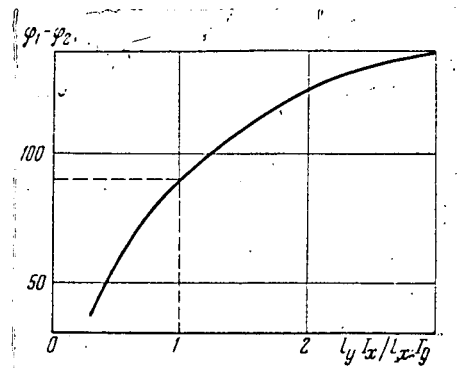
Figure 3

Equations (1.8) and (1.11) imply that in general the control jet engines are not oriented orthogonally to each other, but instead make an angle $\varphi_1 - \varphi_2$ whose tangent is given by

$$\operatorname{tg}(\varphi_1 - \varphi_2) = \frac{2}{\sin 2\alpha_0}\left[\frac{l_y I_x}{l_x I_y}\frac{1}{1 - \left(\frac{l_y I_x}{l_x I_y}\right)^2}\right] \cdot \qquad (1.12)$$

If $l_y I_x / l_x I_y = 1$, then the control jet engines producing the moments u_γ and u_ψ are orthogonal $[(\varphi_1 - \varphi_2) = 90°]$. Figure 3 is an example of the plot of $(\varphi_1 - \varphi_2)$ versus $l_y I_x / l_x I_y$ for $\alpha_0 = 30°$.

Similar transformations can be performed when analyzing the aerodynamic control. It is easy to show that for yaw control u_ψ the moments u_x and u_y must be related by

$$u_x / u_y = \operatorname{tg}\alpha_0, \qquad (1.13)$$

where

$$|u_\psi| = 2|u_y|\frac{1}{\cos\alpha_0} \cdot \qquad (1.14)$$

For bank control, u_x and u_y must be related by

$$u_x | u_y = -1 | \operatorname{tg} \alpha_0, | \qquad (1.15)$$

where

$$| u_\gamma | = 2 | u_x | \frac{1}{\cos \alpha_0}. | \qquad (1.16)$$

For convenience in analysis and to obtain more general results, we shall transform the equations to a dimensionless form. We introduce the dimensionless time τ

$$d\tau = \sqrt{-\sigma\beta}\, dt. | \qquad (1.17)$$

Let us change the scales of the independent variables by taking into account the restriction on the control of u_ψ and u_γ

$$| u_\psi | \leqslant | U_\psi |, \qquad | u_\gamma | \leqslant | U_\gamma |, | \qquad (1.18)$$

introducing for this purpose the following notation

$$
\begin{aligned}
\bar{\beta} &= \beta \frac{-\sigma\beta}{|U_\psi|}, & \overline{\Omega}_\psi &= \frac{\Omega_\psi \sqrt{\sigma\beta}}{|U_\psi|}, \\
\sigma_\gamma^* &= \frac{\sigma_\gamma |u_\psi|}{(-\sigma_\beta)|U_\gamma|}, & \Omega_\gamma &= \frac{\Omega_\gamma \sqrt{-\sigma\beta}}{|U_\gamma|}, \\
\bar{\gamma} &= \frac{\gamma(-\sigma_\beta)}{|U_\gamma|}.
\end{aligned}
\qquad (1.19)
$$

Considering the notation in (1.19), the equations of motion will become

$$
\begin{aligned}
\overline{\Omega}_\psi' &= -\bar{\beta} + \bar{u}_\psi, \\
\bar{\beta}' &= \overline{\Omega}_\psi, \quad |\bar{u}_\psi| \leqslant 1; \qquad (1.20) \\
\overline{\Omega}_\gamma' &= \sigma_\gamma^* \bar{\beta} + \bar{u}_\gamma, \\
\bar{\gamma}' &= \overline{\Omega}_\gamma, |\bar{u}_\gamma| \leqslant 1. \qquad (1.21)
\end{aligned}
$$

We shall now proceed to analyze the optimal control of a spacecraft using Equations (1.20) and (1.21).

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2. An Investigation of the Form of p-Trajectories

In accordance with the Pontryagin maximum principle [7, 8], we write a function

$$H = - p_1 \bar{\beta} + p_1 \bar{u}_\psi - p_2 \overline{\Omega}_\psi + p_3 \overline{\sigma}_\gamma^* \beta + p_3 \bar{u}_\gamma + p_4 \overline{\Omega}_\gamma. \qquad (2.1)$$

Let us separate the terms in H that contain controls

$$H_u = p_1 \bar{u}_\psi + p_3 \bar{u}_\gamma. \qquad (2.2)$$

From the condition that H be maximized relative to u_j, it follows that the controls $\bar{u}_\psi$ and $\bar{u}_\gamma$ are in the form of relay functions that can be found from the condition

$$\bar{u}_\psi = \text{sign } p_1, \quad \bar{u}_\gamma = \text{sign } p_3, \qquad (|\bar{u}_\psi| = |\bar{u}_\gamma| = 1). \qquad (2.3)$$

In order to find the optimal control, it is necessary to find the solution of a system of conjugate equations $p_i' = - \partial H / \partial X_i.$

Let us construct a system of conjugate equations and con- /486

Let us construct a system of conjugate equations and con-
sider the possible forms of solutions to this system. The conjugate equations have the form

$$p_1' = - p_2, \qquad p_2' = p_1 - p_3 \sigma_\gamma^*; \qquad (2.4)$$
$$p_3' = - p_4, \qquad p_4' = 0. \qquad (2.5)$$

The equations in p_3' and p_4' can be easily integrated to

$$p_3 = - c_4 \tau + c_3, \qquad p_4 = c_4. \qquad (2.6)$$

The Solution (2.6) for p_3 (τ) and the Condition (2.3) imply that in general the control of $\bar{u}_\gamma$ may change the sign no more than once.
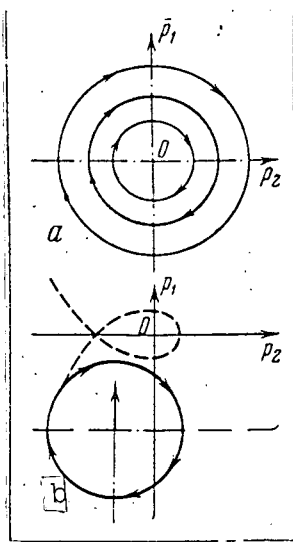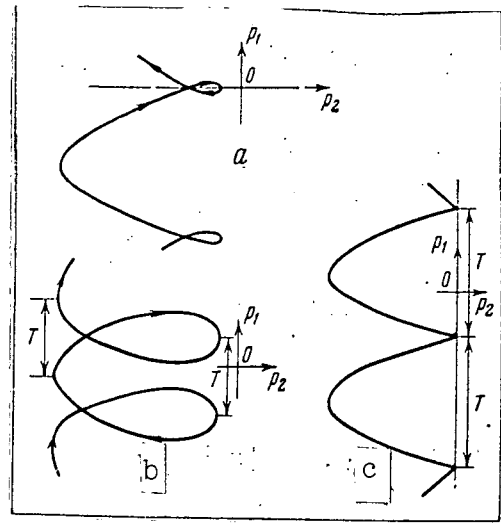
Figure 4   Figure 5

The conjugate variables p_1, ..., p_4 can be determined by taking into account the boundary conditions imposed on the real variables. Since the conjugate equations are linear and the control functions have a relay form, the conjugate variables can be determined up to an arbitrary constant factor. In this connection, (2.4) implies that the form of the solutions for p_1 and p_2 does not depend on the coefficient σ_γ^*, since a change of that coefficient is equivalent to a change of scale of the variable p_3. Only the solution for the real variables depends on the coefficient σ_γ^*.

Let us consider in more detail the solutions for the conjugate variables p_1, p_2. Considering the form of the solution for p_3 (τ), we use a transformation of variables

$$\bar{p}_1 = p_1 - c_3\sigma_\gamma^* + c_4\sigma_\gamma^*\tau, \qquad \bar{p}_2 = p_2 - c_4\sigma_\gamma^*, \tag{2.7}$$

to obtain the equations for $\bar{p}_1$ and $\bar{p}_2$, which can be easily integrated

$$\bar{p}_1' = -\bar{p}_2, \qquad \bar{p}_2' = \bar{p}_1. \qquad\qquad (2.8)$$

A solution of this system of equations in the phase plane $\bar{p}_1\bar{p}_2$ represents a family of concentric circles (Figure 4a).

In the analysis of the optimal control, it is necessary that only the function p_1 (τ) vary with time (the values of the function p_2 are unimportant). Equations (2.7) and (2.8) imply that a change of the variables p_1 and p_2 in the phase plane may /487 be represented as a composition of two motions: the motion of the figurative point along a circle and a translation of the circle (Figure 4b). The possible forms of such curves are illus-trated in Figure 5. Let us note certain properties of our solutions for the conjugate variables (p_1, p_2). It can be shown that the phase curves are symmetric about the axis p_1 = const passing through points at which $\partial p_1/\partial p_2 = 0.$

To prove this, let us consider the projections of the velocity of the figurative point onto the vertical (Op_1) and horizontal (Op_2) axes. These projections are equal (Figure 6):

$$V_{p_1} = -R\cos\varphi + \frac{2a}{T}, \qquad V_{p_2} = R\sin\varphi, \qquad (2.9)$$

where R is the radius of the circle of the p-trajectory; 2a/T is the speed of translation of the center of the circle.

Making use of the expressions for the velocity components V_{p1} and V_{p2}, we obtain the derivative $\partial p_1/\partial p_2$:

$$\frac{\partial p_1}{\partial p_2} = \frac{V_{p_1}}{V_{p_2}} = \frac{-R\cos\varphi + \frac{2a}{T}}{R\sin\varphi}. \qquad (2.10)$$
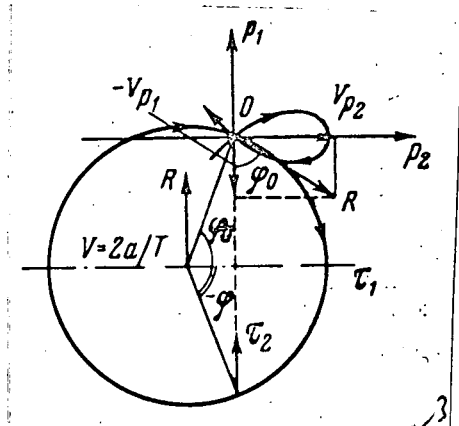

Figure 6

Equation (2.10) implies that the derivatives $\partial p_1/\partial p_2|$ for equal (but different in sign) values of ϕ have equal values that are different in sign. In particular, the derivative $\partial p_1/\partial p_2|$ vanishes if ϕ is given by

$$\varphi = \pm \arccos \frac{2a}{TR},$$ (2.11)

and tends to infinity for $\phi = 0$, and $\phi = 180°$. This type of variation of the derivatives indicates that the phase curve has an axis of symmetry corresponding to the angle $\phi = 0$.

The time spent by the figurative point in commuting between the points of tangency of the phase curve with the generator $p_2 = $ const is equal to the period of the point going around the circle (i.e., 2π), and the distance between the points of tangency is identical for both generators (Figure 5b).

In a region where the direction of motion around the circle and the displacements of the center of the circle are in opposite directions, the phase curve $p_1(p_2)$ may have loops. However, in the case when the speed of translation of the center of the circle is greater than the speed of the figurative point in its movement around the circle, any loops in the phase trajectory disappear.

Let us estimate the time needed for the figurative point /488 to go around a loop (see Figure 6). This can be found from the condition that the time τ_1, during which the figurative point moves along an arc of the circle, be equal to the time τ_2 of the translation motion of a point symmetrically located on the circle.

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We have

$$\tau_1 = 2\varphi_0, \qquad \tau_2 = \frac{R \sin \varphi_0}{a} T.$$

(2.12)

The condition $\tau_1 = \tau_2$ yields an expression for the radius of the circle $\overline{R} = R/a$ in terms of ϕ_0 and T

$$\overline{R} = \frac{2\varphi_0}{T} \frac{1}{\sin \varphi_0}.$$

(2.13)

3. An Analysis of the Time-Optimal Bank Control
of a Spacecraft

We proceed now to analyze the optimal bank control of a spacecraft. The motion of the craft relative to its center of mass will be considered in the phase planes $\overline{\beta\Omega_{\psi}}$ and $\overline{\gamma\Omega_{\gamma}}$. The equations of motion (1.20) imply that, for a proper choice of the control organs, the yawing motion of a spacecraft does not depend on its banking motion, and may be analyzed separately.

Let us consider the yawing motion of the craft

$$\overline{\Omega_{\psi}'} = -\overline{\beta} + \overline{u}_{\psi}, \quad \overline{\beta}' = \overline{\Omega}_{\psi}.$$

(3.1)

For $\overline{u}_{\psi} = 0$ the motion of the craft in the phase plane $\overline{\beta\Omega_{\psi}}$ can be represented by a circle with the center at 0. The radius of the circle depends on the initial conditions $\varphi(0)$ and $\overline{\Omega}_{\psi}(0)$: $R = \sqrt{\overline{\beta(0)^2 + \overline{\Omega}_{\psi}^2(0)}}.$ The figurative point goes clockwise around the circle, making a full revolution during $T_n = 2\pi.$ The motion of the spacecraft in the case of the relay control $\overline{u}_{\psi} = \pm 1$ can be represented in the phase plane by circles with centers moved along the $0\overline{\beta}$ axis by + and − 1, respectively (Figure 7a).
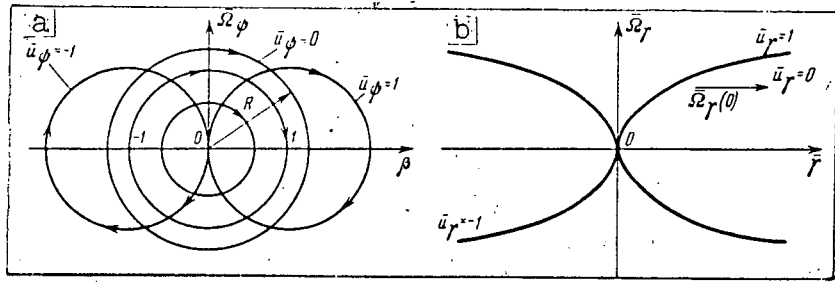
RC

Figure 7

The banking (γ) motion of the craft in general, when $\overset{\cdot}{\sigma_\gamma} \neq 0,$ depends on its yawing motion and on the control $\overline{u}_\gamma$

$$\overline{\Omega}'_\gamma = \sigma^*_i \overline{\beta} + \overline{u}_\gamma, \quad \overline{\gamma}' = \overline{\Omega}_\gamma.$$ (3.2)

In the case when $\overline{\beta} \equiv 0,$ the banking motion of the spacecraft in the phase plane is described by a parabola for $\overline{u}_\gamma \neq 0$ or a straight line $\overline{\Omega}_\gamma = \text{const}$ for $\overline{u}_\gamma = 0$ (Figure 7b).

Let us consider the problem of the time optimization of spacecraft control, assuming that all variables are initially set to zero

$$\overline{\beta}(0) = \overline{\Omega}_\psi(0) = \overline{\Omega}_\gamma(0) = 0, \quad \overline{\gamma}(0) = 0,$$ (3.3)

and such that the craft will be rotated by an angle $\overline{\gamma}_0$ during a minimum time T and will be brought at the end of the turn back to zero conditions

$$\overline{\beta}(T) = \overline{\Omega}_\psi(T) = \overline{\Omega}_\gamma(T) = 0, \quad \overline{\gamma}(T) = \overline{\gamma}_0.$$ (3.4)

Such a system of boundary conditions must be satisfied by analyzing a simultaneous solution of the equations of motion (3.1) and (3.2) for a relay control of $\overline{u}_\psi$ and $\overline{u}_\gamma$, which can be determined taking into account the solutions of the system of conjugate equations (2.4) and (2.5).

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The optimization condition implies that the control of $\overline{U}_\psi$ is linear and determined by the variation of the sign of the conjugate function P_1 (τ). Figure 5 implies that the function p_1 (τ) may either have two segments with different signs (Figure 5c) (two-pulse control), or four segments with different signs (four-pulse control) (Figure 5a, b). Three segments with different signs are also possible.

Let us consider a symmetric bank and yaw control of a spacecraft. An analysis of the actual motion of a spacecraft relative to the angle β implies that, in order for the boundary conditions (3.3) and (3.4) to be satisfied when T is not a multiple of 2π, it is necessary that more than two pulses be applied.

Figure 8 shows an example of the motion of a spacecraft in the phase plane $\overline{\beta}\overline{\Omega}_\psi$, that satisfies the boundary conditions of the problem. It will be shown that it is possible to construct a solution for the conjugate variables p_1 (τ) and p_3 (τ), and consequently the control is indeed optimal.
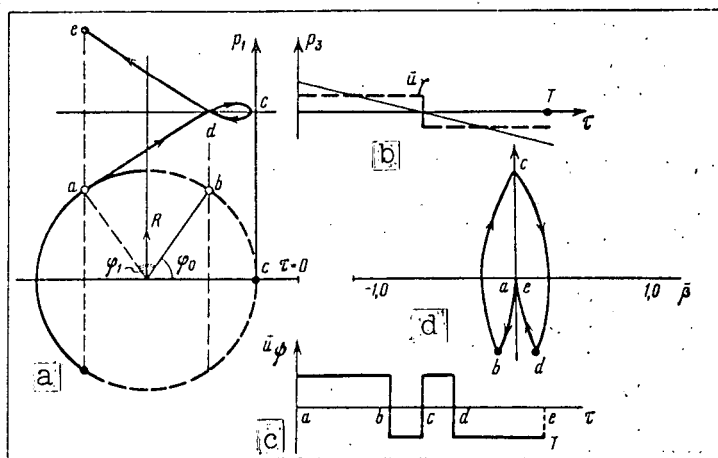

Figure 8

The solution for the conjugate variable p_3 (τ) is a linear function of time. Here we can assume that $p_3(0) = 1/\dot{\sigma}_\gamma$; then

$$p_3(T) = -\frac{1}{\sigma_\gamma^*}, \qquad p_3(\tau) = \frac{1}{\sigma_\gamma^*} - \frac{2\tau}{T\sigma_\gamma^*} \qquad (3.5)$$

The function $p_3(\tau)$ changes sign at $\tau = T/2$, and the control $\bar{u}_\gamma$
changes its sign at the same time (Figure 8b).

By specifying the function $p_3(\tau)$, we determine the location
of the center of the circle p_1 (p_2), in particular $p_1 = -1$
for $\tau = 0$, and $p_1 = +1$ for $\tau = T$.

An analysis of
the actual motion
of a spacecraft also
gives us times at
which the second
pulses (Figure 8c)
are applied. These
pulses correspond to
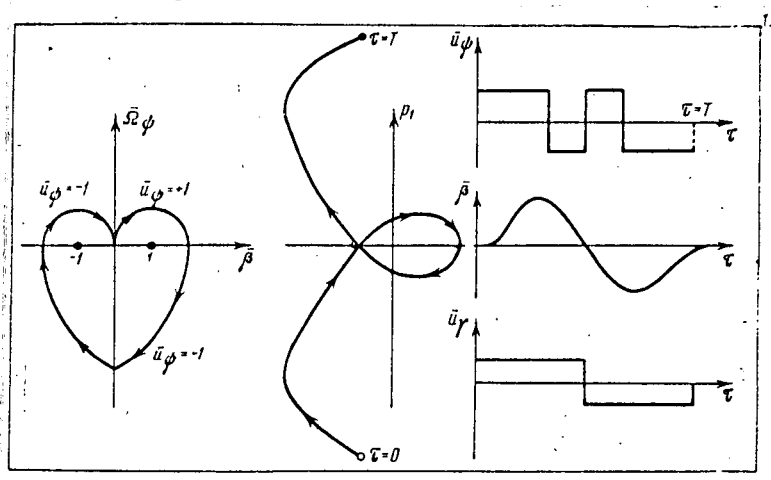a change of the
variable $p_1(\tau)$
along the loop bcd
(Figure 8a). Making
use of Equation (2.13)

Figure 9

and the known time $\tau_1/2 = \varphi_c$, we find the radius of the circle R in
order to solve the conjugate equations. By adding to the angle
ϕ_0, which is proportional to the duration of the second and
third pulses, the angle ϕ_1, which is proportional to the duration
of the first and fourth pulses, we obtain the initial position
of the figurative point on the circle p_1 (p_2).

Using the results obtained in Section 2, it is easy to show
that a change in $P_1(\tau)$ agrees with the required change of the
control $\bar{U}_\psi$, and consequently, the optimal control, satisfying
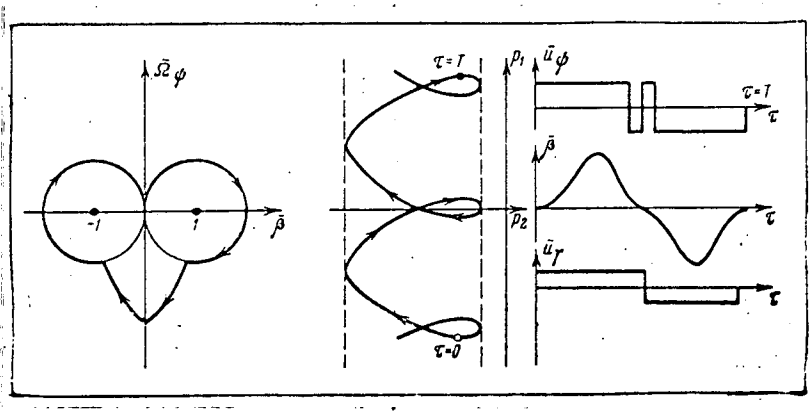the boundary conditions (3.3) and (3.4), has been found.

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Figure 10

As the angle of turn, $\overline{\gamma}_0$, increases, so does the duration of the transition process T, and so do the durations of the second and third pulse. For a certain value of the duration of the transition process T, the dura-
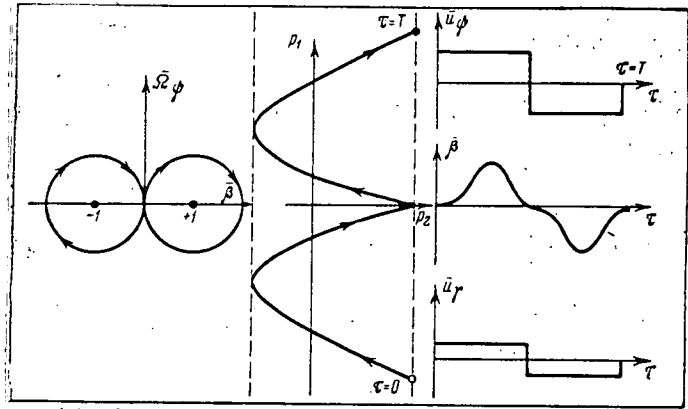

Figure 11

tions of the second and third pulse are maximum, and begin to decrease as T continues to increase (Figures 9 and 10). For $\overline{\gamma}_0$, corresponding to $\overline{\beta}(\tau, T)$ (where n is an integer), the control is an extremum and is accomplished with two pulses (Figure 11). When $\overline{\gamma}_0$ continues to increase (increase of T), the control is again accomplished with four pulses (Figure 12).

An analysis of the three-pulse control (Figure 13), satisfying the boundary conditions (3.3) and (3.4), shows that — even though this control is feasible — it is not time-optimal.
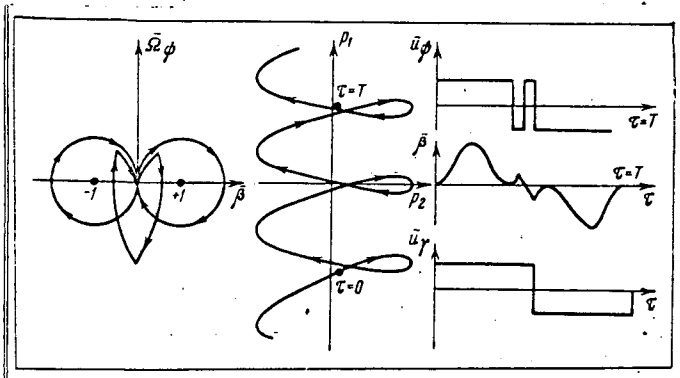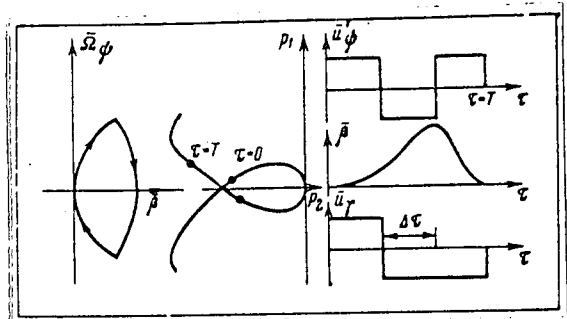
Figure 12

Figure 13

This is due to the fact that, in addition to an additional in-
crease of $\overline{\gamma}_0$ due to $\overline{\beta}$, also the banking angular velocity is de-
veloped, whose compensation results in a loss of the gain obtained
and deprives the motion of its optimality. In all the above
cases (Figures 8 - 12), the motion of the spacecraft along the
angle $\overline{\beta}$ is symmetric, i.e.,

$$\int_0^{T/2} \overline{\beta}\,d\tau = -\int_{T/2}^{T} \overline{\beta}\,d\tau. \qquad (3.6)$$

As a result, the additional banking angular velocity within
the time interval [0, T/2], due to the slippage, is compensated
within the time interval (T/2, T) by a further increase in $\overline{\beta}$.
Due to this fact, the time intervals during which the positive
and negative control $\overline{U}_\psi$ is exerted are identical.

In order to find a relationship between $\overline{\gamma}_0$ and the minimum
duration of the transition process T, let us transform Equations
(3.2). Let us represent $\overline{\Omega}_\gamma$ and $\overline{\gamma}$ as sums of two terms

$$\overline{\Omega}_\gamma = \overline{\Omega}_{\gamma_1} + \overline{\Omega}_{\gamma_2}, \quad \gamma = \gamma_1 + \gamma_2, \qquad (3.7)$$

and let the variables $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\Omega}_{\gamma_1}$, and $\bar{\Omega}_{\gamma_2}$ be found from the equations

$$\bar{\Omega}'_{\gamma_1} = \dot{\sigma}_\gamma \bar{\beta}, \quad \bar{\gamma}'_1 = \bar{\Omega}_{\gamma_1}; \tag{3.8}$$
$$\bar{\Omega}'_{\gamma_2} = \bar{u}_\gamma, \quad \bar{\gamma}'_2 = \bar{\Omega}_{\gamma_2}. \tag{3.9}$$

Equations (3.8) imply that

$$\bar{\Omega}_{\gamma_1}(T) = \dot{\sigma}_{\gamma_0} \int_0^T \bar{\beta}\,d\tau, \quad \bar{\gamma}_1(T) = \dot{\sigma}_\gamma \int_0^T \int \bar{\beta}\,d\tau\,d\tau. \tag{3.10}$$

Above, an analysis of $\bar{\beta}(\tau)$ has shown that $\int_0^T \bar{\beta}\,d\tau = 0$. As a result, Equations (3.9) may be solved separately, without considering Equations (3.8). Thus the solution for $\bar{\gamma}_0(T)$ can be written in the form

$$\bar{\gamma}_0(T) = 2\left(\frac{T}{2}\right)^2 + \dot{\sigma}_\gamma \int_0^T \int \bar{\beta}(\tau, T)\,d\tau\,d\tau \frac{T^2}{2} + \gamma_2. \tag{3.11}$$

The Relation (3.11) permits us to find the dependence of $\bar{\gamma}_0$ on the duration of the transition process T.

The function $\bar{\beta}(\tau, T)$, appearing in the relation, depends on the time T since the control $\bar{u}_\psi$ also depends strongly on it.

Using Equation (3.11), we can easily find the final value of the bank angle $\bar{\gamma}_0$ that corresponds to the two-pulse control (Figure 11)

$$\bar{\gamma}_0 = 4\pi^2(2 + \dot{\sigma}_\gamma). \tag{3.12}$$

In general, if the time T is a multiple of 4π, then the expression for $\bar{\gamma}_n$ can be written in the form

$$\bar{\gamma}_n = (2n)^2 \pi^2 (2 + \dot{\sigma}_\gamma), \tag{3.13}$$

where n is the number of periods of the natural oscillations in $\bar{\beta}$ within one half of the time interval during which the transition process takes place.

In spite of the simplicity of the equations of motion, finding $\bar{\gamma}_0(T)$ is a very difficult operation. Let us write down certain necessary relations. An analysis of the geometry of motion in the phase plane (Figure 14) gives the following expressions for the basic parameters:

$$R = \sqrt{5 - 4\cos\varphi_1}, \qquad (3.14)$$

$$\varphi_2 = \text{arctg} \; \frac{\dfrac{\sin\varphi_1}{2 - \cos\varphi_1} + 2\sqrt{1 - \cos\varphi_1}}{1 - \dfrac{2\sin\varphi_1}{2 - \cos\varphi_1}\sqrt{1 - \cos\varphi_1}}, \qquad (3.15)$$

where ϕ_1 is the angle which is proportional to the duration of the first (fourth) pulse; ϕ_2 is the angle which is proportional to the duration of the second (third) pulse.

Figure 15 gives the plot of $\overline{\varphi_2(\varphi_1)}$, and Figure 16 the plot of $\overline{\gamma_2/2\sigma_\gamma}$ versus ϕ_1. Using these plots we can find $\bar{\gamma}_0(T)$ for each value of σ_γ^*.
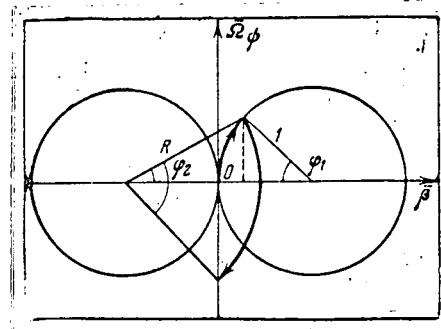

Figure 14

Let us estimate the gain in time ($\Delta\tau$), due to the simultaneous bank and yaw control as compared with the optimal control of the bank angle alone. The time $(T_0 + \Delta\tau)$ needed to turn a spacecraft by an angle $\bar{\gamma}_1 + \bar{\gamma}_2$ in the case of an optimal control of the bank angle alone is given by

$$\gamma_1 + \gamma_2 = 2\left(\frac{T_0 + \Delta\tau}{2}\right)^2, \qquad (3.17)$$

Figure 15

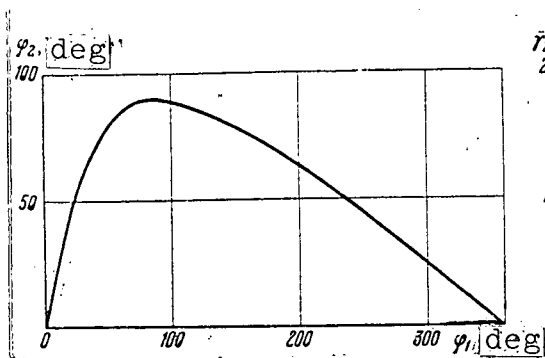

Figure 16
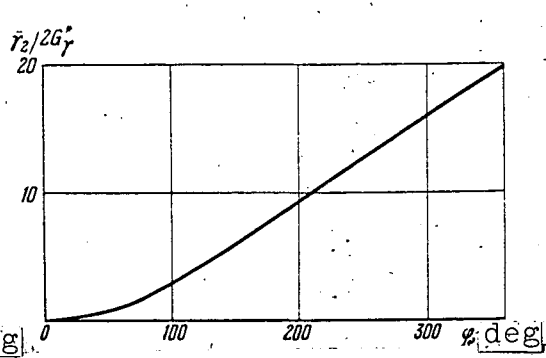
where γ_1 is the bank angle achieved as a result of bank control
during a time-optimal maneuver performed during a time T_0; γ_2
is the bank angle obtained during a time optimal slippage maneu-
ver; T_0 is the duration of the maneuver in the case of the time-
optimal bank and yaw control.

From (3.17), we obtain a nonlinear dependence of the time
gain on T_0 and γ_2

$$\Delta\tau = -T_0 + \sqrt{T_0^2 + 2\gamma_2}. \tag{3.18}$$

4.   Time-Optimal Compensation of the Initial Yaw and
Angle-of-Attack Deflections of a Spacecraft

Let us consider the problem of the time optimization of the
compensation for the yaw deviation of a spacecraft.   As before,
we shall analyze the equations of motion

$$\bar{\Omega}_\psi' = -\bar{\beta}' + \bar{u}_\psi, \quad \bar{\beta}' = \bar{\Omega}_\psi, \tag{4.1}$$
$$\bar{\Omega}_\gamma' = \sigma_\gamma \dot{\bar{\beta}} + \bar{u}_\gamma, \quad \bar{\gamma}' = \bar{\Omega}_\gamma \tag{4.2}$$

with the boundary conditions

$$\bar{\Omega}_\psi(0) = 0, \; \bar{\beta}(0) = \bar{\beta}_0, \; \bar{\Omega}_\gamma(0) = \bar{\gamma}(0) = 0,$$
$$\bar{\Omega}_\psi(T) = \bar{\beta}(T) = \bar{\Omega}_\gamma(\tau) = \bar{\gamma}(T) = 0. \qquad (4.3)$$

The equations of motion (4.1) and (4.2) imply that the motion along the angle $\bar{\beta}$ may be discussed separately from the motion along the angle $\bar{\gamma}$. Thus, the problem of compensating for the initial deflection in the angle $\bar{\beta}$ can be subdivided into the problem of the optimal yaw control of the craft and the problem of compensating for the accumulated bank error.

The solution of the first problem is known; it is considered in a number of papers. Moreover, its solution has been carried to a stage involving a synthesis of a system realizing the optimal control algorithm [7].

However, the complete problem of the optimal control of $\bar{\beta}$ and $\bar{\gamma}$ has not yet apparently been studied.

It should also be noted that the problem of the time-optimal compensation of a deviation in α coincides with the problem of compensating a deviation in β [see Equations (1.1) and (1.2)], and its solution, as noted above, is known.

It is easy to show that the problem of the optimal compensation of deviations in $\bar{\beta}$ and $\bar{\gamma}$ does not have a unique solution. This is due to the fact that the optimal control of $\bar{\beta}$ does not depend on the motion of the spacecraft along the angle $\bar{\gamma}$ and has its "characteristic" time of the duration of the transition process. At the same time, a control of $\bar{\gamma}$ depends on the yawing motion of the spacecraft. In the case when the bank control is of little effectiveness and the transition process in $\bar{\gamma}$ is slower than the one in $\bar{\beta}$, the control is unique. In the case when the effectiveness of $\bar{U}_\gamma$ is high, deviations in the angle $\bar{\gamma}$

may be compensated for in many different ways provided that the process stops at a time determined by the motion along $\bar{\beta}$. Examples of those types of motion are illustrated in Figure 17.

Let us consider in somewhat greater detail the case when the solution of the problem is unique — namely, when the control of $\bar{\gamma}$ is of little effectiveness and the process of compensating for perturbations in $\bar{\gamma}$ due to the hawing motion of the spacecraft takes a longer time than the process of compensating for the motion along $\bar{\beta}$.

Similarly to Equation (3.7), let us represent a change in $\bar{\gamma}$ as consisting of two components: "forced", ($\bar{\gamma}_1$), caused by a perturbation in $\bar{\beta}$, and "compensating", ($\bar{\gamma}_2$), due to the bank control

$$\bar{\gamma}_1' = \bar{\Omega}_{\gamma_1}, \qquad \bar{\Omega}_{\gamma_1}' = \sigma_\gamma^* \bar{\beta}; \qquad (4.4)$$

$$\bar{\gamma}_2' = \bar{\Omega}_{\gamma_2}, \qquad \bar{\Omega}_{\gamma_2}' = \bar{u}_\gamma. \qquad (4.5)$$

The solution for $\bar{\gamma}$ has the form

$$\bar{\gamma} = \bar{\gamma}_1 + \bar{\gamma}_2. \qquad (4.6)$$

Since in this case the transition process involving β ends /495 earlier than the process involving $\bar{\gamma}$ $(T_\beta < T_\gamma)$, the solution of Equations (4.4) can be used as the initial conditions for Equations (4.5), and be written in the form

$$\bar{\gamma}_2(0) = \sigma_\gamma^* \left(\int_0^{T_\beta} \int \bar{\beta} d\tau d\tau - T_\beta \int_0^{T_\beta} \bar{\beta} d\tau \right) = \sigma_\gamma^* (I_2 - T_\beta I_1), \qquad (4.7)$$

$$\bar{\gamma}_2'(0) = \sigma_\gamma^* \int_0^{T_\beta} \bar{\beta} d\tau = \sigma_\gamma^* I_1, \qquad (4.8)$$

where $I_1 = \int_0^{T_\beta} \bar{\beta} d\tau, \quad I_2 = \int_0^{T_\beta} \bar{\beta} d\tau d\tau$.
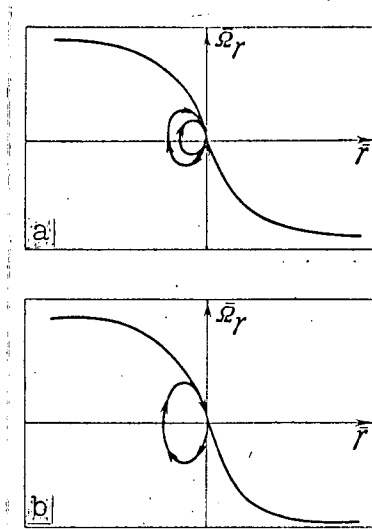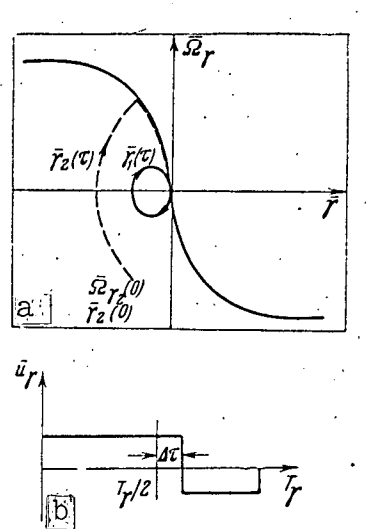
Figure 17

Figure 18

In Equation (4.7) the integral I_1 accounts for the varia-
tion of $\bar{\gamma}$ within the time interval $(0, T_\beta)$, due to the initial
conditions on γ_2' (0) (4.8). Thus, the problem has been reduced
to the well-known problem of an independent bank control of the
motion of a spacecraft. An example of a phase trajectory is
given in Figure 18a, which gives the plots of $\bar{\gamma}_2$ (τ) and $\bar{\gamma}_1$ (τ).

As we know, the time-optimal bank control is achieved by
means of two pulses (Figure 18b) whose duration differs by $\overline{\Delta\tau}$.
The latter can be determined from the condition that the initial
angular velocity $\bar{\gamma}_2'$ (0) be compensated for:

$$2\Delta\tau = \bar{\gamma}_2'(0), \qquad \Delta\tau = \frac{I_1 \sigma_\gamma^*}{2}. \qquad (4.9)$$

The total duration of the bank transition process T_γ is
found from the condition that the total bank deviation, due
to a motion in $\bar{\beta}$, be compensated for:

$$\gamma_2^\Sigma = \dot{\sigma}_\gamma \left[I_2 + (T_\gamma - T_\beta) I_1 \right] \qquad (4.10)$$

T_γ can be found from

$$T_\gamma = 2 \left[\Delta\tau + \sqrt{2(\Delta\tau)^2 + \gamma_2} \right] \qquad (4.11)$$

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